CHEM-E7225 Advanced Process Control D Winter 2026

Lecture 08 – Model Predictive Control: Noise/Disturbances

Otacilio "Minho" Neto (otacilio.bneto@pm.me)

The previous lectures have established the foundations of designing model-predictive controllers. However, our design was under the assumption that the process state evolves as predicted and could be perfectly measured. In this lecture, we (finally!) formalize the effects of disturbances/noise on dynamical systems and on our ability to observe them.

1 Process disturbances and measurement noise

A controlled dynamical system subject to external noise/disturbances can be represented by the state-space model

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t), u(t), w(t)); \tag{1a}$$
$$y(t) = g(x(t), u(t), v(t)), \tag{1b}$$

where, in this case, the dynamics $f(\cdot)$ are described in terms of the *state* $x(t) \in \mathbb{R}^{N_x}$, the *controls* $u(t) \in \mathbb{R}^{N_u}$, and the *process disturbances* $w(t) \in \mathbb{R}^{N_w}$. Additionally, we lift the assumption that the state $x(t)$ is perfectly measurable and introduce the *measurement signal* (or *output signal*) $y : \mathbb{R} \to \mathbb{R}^{N_y}$. The function $g(\cdot)$ describes how the measurements $y(t) \in \mathbb{R}^{N_y}$ are formed based on the state $x(t)$, the controls $u(t)$, and the *measurement noise* $v(t) \in \mathbb{R}^{N_v}$. While the state equation (Eq. 1a) is a differential equation, the measurement equation (Eq. 1b) is just an algebraic equation. The block diagram of a partially-observed system subject to external noise/disturbances is shown in Figure 1. In practice, the measurements $y(t)$ rarely depend on the controls $u(t)$—that is, *input feedthrough* is not common. For this reason, we will proceed by considering measurement models of the form $y(t) = g(x(t), v(t))$.

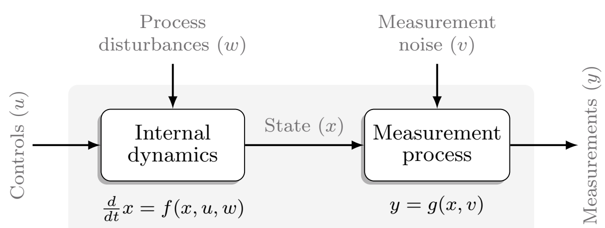

Figure 1. A partially-observed noisy system.

The noise/disturbances $(v(\cdot), w(\cdot))$ act as inputs, in the sense that they are external forces affecting the dynamics and measurement process. However, they are different from control inputs in that *their future values cannot be chosen or anticipated*. As a consequence, the common modelling assumption is to treat $w(\cdot)$ and $v(\cdot)$ as *random processes*, described by their mean (i.e., their expected value) and autocorrelation (i.e., a time-dependent notion of variance) [1].

Example 1.1 ***Isothermal CSTR with feed disturbances and noisy measurements*** [2]

A CSTR implements the isothermal process $A \xrightarrow{\kappa} B$ as in Figure 2. The inlet flowrate $F(t)$ [L/min] can be manipulated, but its concentrations $c_{Ai}(t)$ [mol/L] *cannot*. The reactor is equipped with a sensor providing real-time (noisy) measurements of the product $c_A(t)$ [mol/L].

The dynamics and measurements of this reactor are modelled as

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = u(t)(w(t) - x(t)) - \kappa x(t); \tag{2a}$$
$$y(t) = x(t) + v(t), \tag{2b}$$

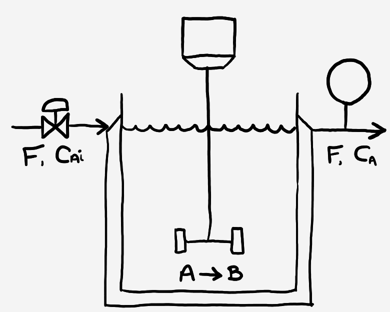

Figure 2

with state $x(t) = c_A(t)$, controls $u(t) = \frac{1}{V}F(t)$, disturbances $w(t) = c_{Ai}(t)$, and measurement noise $v(t)$. When $w(\cdot)$ and $v(\cdot)$ are unknown, they are commonly modelled as random processes with known means ($\mu_w = \mathrm{E}w(t)$ and $\mu_v = \mathrm{E}v(t)$) and autocorrelations ($R_w(\tau) = \mathrm{E}[w(t)w(t+\tau)]$ and $R_v(\tau) = \mathrm{E}[v(t)v(t+\tau)]$). The state-space (Eq. 2) with random noise/disturbances is a stochastic differential equation model.

2 (Discrete-time) stochastic linear models

In the following, for the sake of simplicity, we assume that the systems are not controlled (that is, $u(t) = 0$).

An important class of stochastic models are discrete-time linear models represented as

$$x[n+1] = Ax[n] + w[n], \qquad w[n] \sim \mathcal{N}(\mu_w, S_w^2); \tag{3a}$$

$$y[n] = Cx[n] + v[n], \qquad v[n] \sim \mathcal{N}(\mu_v, S_v^2), \tag{3b}$$

given means (μ_w, μ_v) and covariance matrices (S_w^2, S_v^2) of appropriate sizes. The linear model with *additive white Gaussian noise (AWGN)*, despite not the most general, has attractive computational properties and still covers (or approximates) many relevant problems. Using standard properties of Gaussian variables, we can rewrite this model as

$$\begin{cases} x[n+1] = Ax[n] + w[n] \\ \quad\; y[n] = Cx[n] + v[n] \end{cases} \xrightarrow[\square[n]=S_\square\delta_\square[n]+\mu_\square]{\text{Using the fact}} \begin{cases} x[n+1] = Ax[n] + S_w\delta_w[n] + \mu_w \\ \quad\; y[n] = Cx[n] + S_v\delta_v[n] + \mu_v \end{cases} \tag{4}$$

where $\delta_w[n], \delta_v[n] \sim \mathcal{N}(0, I)$ are standard Gaussian variables of appropriate size. This state-space realization (Eq. 4) implies that a linear model with AWGN is equivalent to an affine model with some zero-mean random perturbation. The standard Gaussian distribution is so important that most software tools have specialized functions for their computation; for instance, the command `randn`(N, M) in many programming languages generate a $N \times M$ matrix of samples from this distribution, with `randn`$(N, 1)$ being the command to generate a single N-dimensional sample.

The statistics (μ_w, Σ_w) summarize how the disturbances affect the the system. The *process drift* μ_w represents a constant perturbation to its dynamics; for instance, the nominal concentration of a non-manipulated feedstock, or a constant leakage in a tank reactor. The *square-root covariance* Σ_w quantifies the intensity of these perturbations–large perturbations become more likely when Σ_w is large (that is, when its eigenvalues are large). Similarly, the *sensor drift* μ_v and *square-root covariance* Σ_v summarize the noise affecting the measurement process, which often reflect sensor miscalibration and/or faults. Figure 3 shows the effects of these parameters for a scalar system.

When (μ_w, μ_v) represent drifts caused by physical faults, it is perhaps a better idea to fix these faults through process maintenance rather than through feedback control.

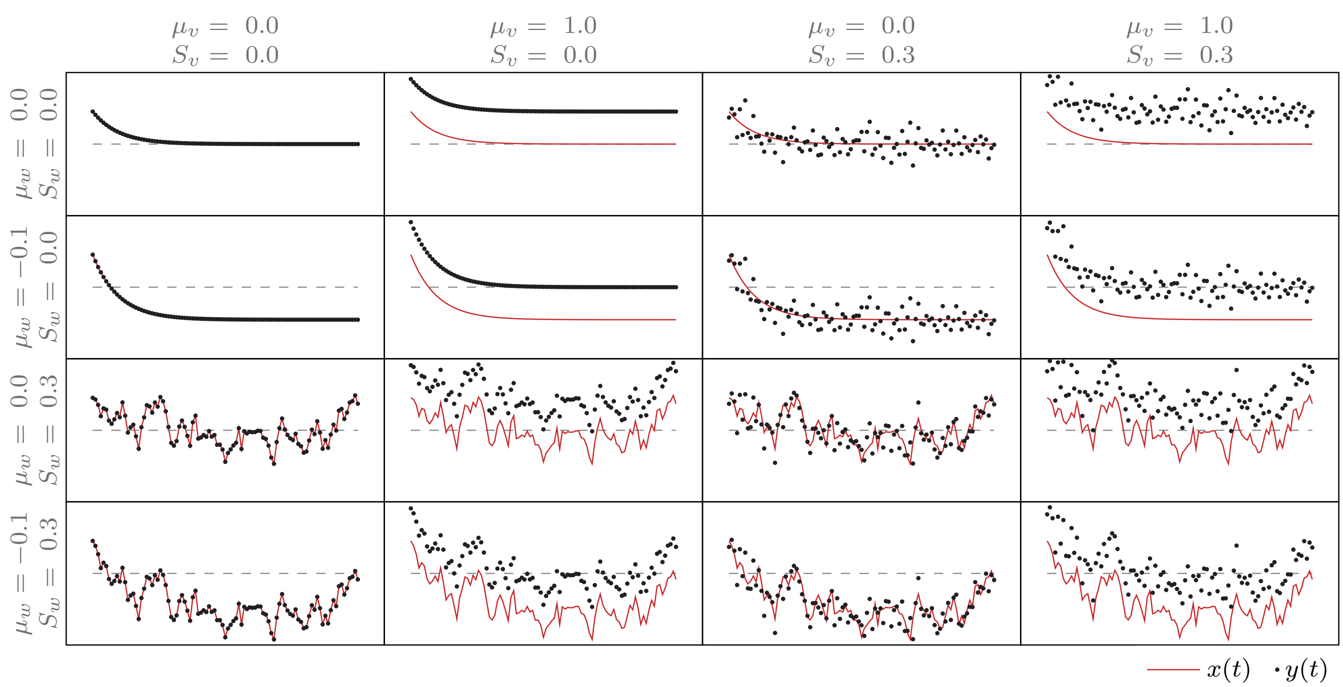

Figure 3. Simulation of the linear dynamics $x[n+1] = 0.9x[n] + w[n]$ and measurement process $y[n] = x[n] + v[n]$ given realizations of the disturbances $w[n] \sim \mathcal{N}(\mu_w, S_w^2)$ and noise $v[n] \sim \mathcal{N}(\mu_v, S_v^2)$ with different parameter values. The initial state is $x[0] = 1$.

Differently from the deterministic case, in a stochastic model *we cannot predict the future states exactly*. Specifically, if $x[n]$ is known, then it is possible to show that $x[n{+}k] \sim \mathcal{N}(\mu_x[k], S_x^2[k])$ for every $k > 0$, with

$$\mu_x[k] = A^k x[n] + \sum_{i=0}^{k-1} A^i \mu_w \qquad \text{and} \qquad S_x^2[k] = \sum_{i=0}^{k-1} (A^i S_w)(A^i S_w)^\mathsf{T}. \tag{5}$$

In other words, the future states are random variables whose uncertainty (as quantified by $S_x^2[\cdot]$) accumulates in time. This holds true even if we measure the state $x[n]$ at every $n > 0$, with the only advantage being that in this case the uncertainty would be time-invariant, since $x[n{+}1] \sim \mathcal{N}(Ax[n] + \mu_w, S_w^2)$ at every prediction step. Thus, in stochastic settings, we must rely on *data-based estimates of the state signal* (e.g., its expected value). Determining past/current states based on data is known as *estimation* or *filtering* (as in *"filtering out the noise"*), while predicting future values of the state is known as *forecasting*. We will discuss state estimation in more details in the next lecture.

2.1 Colored noise

The framework of linear models with AWGN is still general enough to cover a wider variety of systems, including those affect by noise which itself is dynamic. These classes of noise/disturbance processes are known as *colored noise*. We focus on colored noise consisting of processes $d[\cdot]$ evolving according to the *noise dynamics*

$$d[n{+}1] = A_w d[n] + w[n] \tag{6}$$

given some matrix A_w and white Gaussian noise $w[\cdot]$. Based on the matrix A_w, we can distinguish three types of noise: (i) additive white Gaussian noise ($A_w = 0$); (ii) stationary autoregressive noise ($A_w \prec I$); and (iii) nonstationary random walk ($A_w \sim I$). We remark that $A_w \prec I$ means that all the eigenvalues of A_w are smaller than 1, and $A_w \sim I$ means that all its eigenvalues are exactly 1. Figure 4 illustrates these random processes.

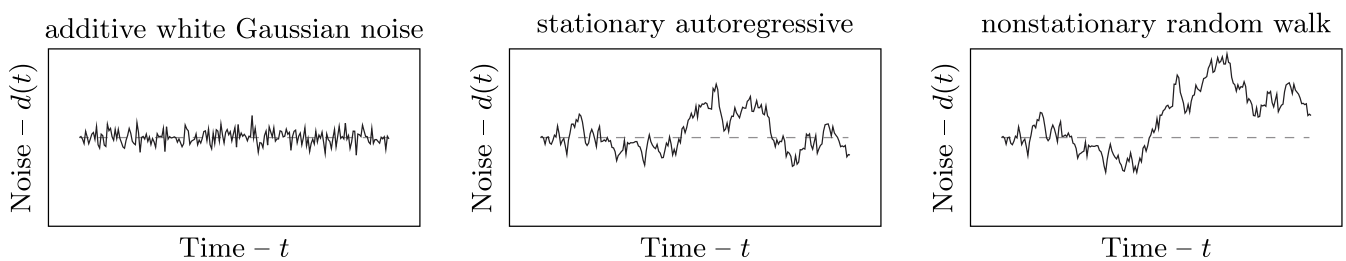

Figure 4. Colored noise for different values of the noise dynamics matrix A_w. For $A_w = 0$, we recover the standard AWGN process. For $A_w \prec I$, we obtain the stationary autoregressive process, in which the noise has some dynamics but is still centered on its mean. For $A_w \sim I$, we obtain the nonstationary random walk process, in which the noise evolves freely and does not have a constant mean value.

The treatment of linear systems subject to colored noise uses our familiar method of *state augmentation*. In this case, we augment the state as $\tilde{x}[n] = (x[n], d[n])$ and rewrite the dynamics as

$$\begin{cases} x[n{+}1] = Ax[n] + d[n] \\ d[n{+}1] = A_w d[n] + w[n] \end{cases} \xrightarrow{\text{State augmentation}} \underbrace{\begin{bmatrix} x[n{+}1] \\ d[n{+}1] \end{bmatrix}}_{\tilde{x}[n+1]} = \underbrace{\begin{bmatrix} A & I \\ & A_w \end{bmatrix}}_{A^d} \underbrace{\begin{bmatrix} x[n] \\ d[n] \end{bmatrix}}_{\tilde{x}[n]} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{B^d} w[n], \tag{7}$$

or, more compactly, $\tilde{x}[n{+}1] = A\tilde{x}[n] + \tilde{w}[n]$ with $\tilde{w}[n] \sim \mathcal{N}(B^d \mu_w,\ B^d S_w^2 {B^d}^\mathsf{T})$. We then proceed using this augmented model as our representation of the dynamics. A similar procedure can be done for the measurement equation. Finally, we remark that the covariance matrix of $\tilde{w}[n]$ ($S_{\tilde{w}}^2 = B^d S_w^2 {B^d}^\mathsf{T}$) is singular, and this might lead to numerical issues. Consequently, in such models, it is common to add a *small* zero-mean AWGN directly to the state-evolution $x[n{+}1]$, which ultimately results in the matrix $B^d = \texttt{col}(\varepsilon I, I)$ for some small parameter $\varepsilon > 0$.

References

[1] A. Papoulis and S. U. Pillai. *Probability, random variables, and stochastic processes.* 4th ed. McGraw-Hill, 2002.

[2] D. E. Seborg et al. *Process dynamics and control.* John Wiley & Sons, 2016.