

Homework 04 – Optimal Control of Process Systems (w/ Constraints)

Otacilio “Minho” Neto (otacilio.bneto@pm.me)

Problem Setup

Let's consider again the *Three Tank System* (Figure 1) from Homework 03. The process consists of three cylindrical tanks (T_i , $i = 1, 2, 3$) connected by two fixed valves (V_i , $i = 1, 2$), with an outflow valve V_0 for the last tank. The task is to control the liquid levels (h_i , $i = 1, 2, 3$) in each tank by manipulating the pumps P_1 and P_3 , respectively. Using mass balance equations, the dynamics of the liquid levels (h_i , $i = 1, 2, 3$) are modelled as

$$S_T \frac{d}{dt} h_1 = Q_1 - f_{12}; \quad (1a)$$

$$S_T \frac{d}{dt} h_2 = f_{12} - f_{23}; \quad (1b)$$

$$S_T \frac{d}{dt} h_3 = Q_3 + f_{23} - f_{30}; \quad (1c)$$

with $f_{ij} = \alpha_V S_V \tanh(h_i - h_j) \sqrt{2g|h_i - h_j|}$ being the liquid flow from T_i to T_j , and $f_{30} = \alpha_0 S_V \sqrt{2gh_3}$ being the outflow from the last tank. The terms $(S_T, S_V, \alpha_V, \alpha_0, g)$ are constant model parameters (Table 1).

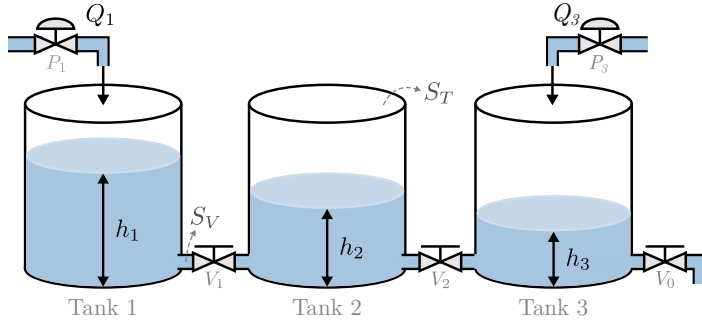


Figure 1. Layout of the Three Tank System.

Symbol	Description [Units]	Value
h_i	Water level of T_i [cm]	$\in [0, 60]$
Q_i	Flow-rate to T_i [ml/s]	$\in [0, 140]$
S_T	Cross-section of T_i [cm ²]	154
S_V	Cross-section of V_i [cm ²]	0.5
α_V	Flow coefficient of V_i [-]	0.47
α_0	Outflow coefficient of V_0 [-]	0.77
g	Gravitational constant [cm/s ²]	981

Table 1. Process variables and model constant parameters.

In control notation, the state signal is $x = (x_1, x_2, x_3) = (h_1, h_2, h_3)$ and the control signal is $u = (u_1, u_2) = (Q_1, Q_3)$. Rewrite the model Eq. (1a)-(1c) in this notation to have it in the form $\frac{d}{dt}x = f(x, u)$, then solve the following tasks:

Task 1. (10 points) Design a L-MPC for controlling the Three Tank System towards the steady-state $(x_{\text{ref}}, u_{\text{ref}})$,

$$x_{\text{ref}} = [50 \ 30 \ 10]^T \quad \text{and} \quad u_{\text{ref}} = [46.551 \ 7.376]^T, \quad (2)$$

while enforcing the input bound constraints $0 \preceq u(t) \preceq 140$ from Table 1. Consider $Q = Q_f = \text{diag}([1 \ 1 \ 1])$, $R = \text{diag}([0.01 \ 0.01])$, $S = 0$, control horizon $T = 30$ [s], and control frequency of $1/\Delta t = 1\text{Hz}$.

Simulate the process under the control policy $u[n] = u_{\text{ref}} + K_{\text{RHC}}(x[n] - x_{\text{ref}})$, given the initial state $x_0 = (10, 30, 50)$ and simulation horizon $T_{\text{sim}} = 240$ [s]. **Comment on the results.** Does the process reach the setpoints $(x_{\text{ref}}, u_{\text{ref}})$? Does it violate flow-rate/water-level limits (see Table 1)?

Task 2. (10 points) Using slack variables, modify the L-MPC from **Task 1** to include the *desirable* state bound constraints $20 \preceq x(t) \preceq 40$ (representing some kind of “*safety region*”). Simulate the process again using this modified L-MPC policy. **Comment on the results.** Does the process reach the setpoints $(x_{\text{ref}}, u_{\text{ref}})$? Is the controller able to enforce the state constraints?

Hint: Remember that the constraints in deviation variables become $0 \preceq u^\delta(t) + u_{\text{ref}} \preceq 140$ and $20 \preceq x^\delta(t) + x_{\text{ref}} \preceq 40$.