

Homework 03 – Optimal Control of Process Systems

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Problem Setup

The *Three Tank System* (Figure 1) is a benchmark process representative of many industrial control problems (e.g., liquid level control in petrochemical plants). It consists of three cylindrical tanks (T_i , $i = 1, 2, 3$) connected by two fixed valves (V_i , $i = 1, 2$), with an outflow valve V_0 for the last tank. We are interested in controlling the liquid levels (h_i , $i = 1, 2, 3$) in each tank by manipulating the incoming flow-rates to tanks T_1 and T_3 through the pumps P_1 and P_3 , respectively. Using mass balance equations, the dynamics of the liquid levels (h_i , $i = 1, 2, 3$) are modelled as

$$S_T \frac{d}{dt} h_1 = Q_1 - f_{12}; \quad (1a)$$

$$S_T \frac{d}{dt} h_2 = f_{12} - f_{23}; \quad (1b)$$

$$S_T \frac{d}{dt} h_3 = Q_3 + f_{23} - f_{30}; \quad (1c)$$

with $f_{ij} = \alpha_V S_V \tanh(h_i - h_j) \sqrt{2g|h_i - h_j|}$ being the liquid flow from T_i to T_j , and $f_{30} = \alpha_0 S_V \sqrt{2gh_3}$ being the outflow from the last tank. The terms $(S_T, S_V, \alpha_V, \alpha_0, g)$ are constant model parameters (Table 1).

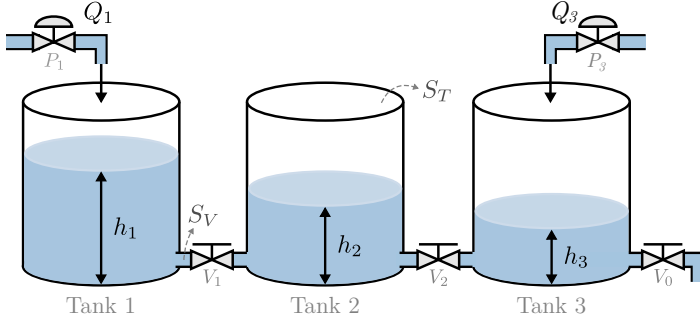


Figure 1. Layout of the Three Tank System.

Symbol	Description [Units]	Value
h_i	Water level of T_i [cm]	$\in [0, 60]$
Q_i	Flow-rate to T_i [ml/s]	$\in [0, 140]$
S_T	Cross-section of T_i [cm ²]	154
S_V	Cross-section of V_i [cm ²]	0.5
α_V	Flow coefficient of V_i [-]	0.47
α_0	Outflow coefficient of V_0 [-]	0.77
g	Gravitational constant [cm/s ²]	981

Table 1. Process variables and model constant parameters.

In control notation, the state signal is $x = (x_1, x_2, x_3) = (h_1, h_2, h_3)$ and the control signal is $u = (u_1, u_2) = (Q_1, Q_3)$. Rewrite the model Eq. (1a)-(1c) in this notation to have it in the form $\frac{d}{dt}x = f(x, u)$, then solve the following tasks:

Task 1. (10 points) Using the linearization method, formulate an unconstrained linear-quadratic regulator problem that represents the task of controlling the Three Tank System towards the steady-state $(x_{\text{ref}}, u_{\text{ref}})$, with

$$x_{\text{ref}} = [50 \ 30 \ 10]^T \quad \text{and} \quad u_{\text{ref}} = [46.551 \ 7.376]^T. \quad (2)$$

Consider diagonal tuning matrices, with $Q = Q_f = \text{diag}([1 \ 1 \ 1])$, $R = \text{diag}([0.01 \ 0.01])$, and $S = 0$, and a control horizon of $T = 30$ [s] (convert to timesteps using a control frequency $1/\Delta t = 1\text{Hz}$).

Task 2. (10 points) Simulate the process under the control policy $u[n] = u_{\text{ref}} + K_{\text{RHC}}(x[n] - x_{\text{ref}})$, given initial state $x_0 = [20 \ 10 \ 20]^T$ and simulation horizon $T_{\text{sim}} = 120$ [s]. **Comment on the results.** Does the system reach the setpoint? Does the control violate process limits (Table 1)? What if you set $R = \text{diag}([100 \ 0.01])$?

Hint: Use a linearized-then-discretized model $x^\delta[n+1] = A_{\Delta t}x^\delta[n] + B_{\Delta t}u^\delta[n]$ for designing the LQR problem, but then use the actual nonlinear integrator $x[n+1] = f_{\Delta t}(x[n], u[n])$ (e.g., RK4) to simulate the system.