On the controllability of activated sludge plants

Otacílio B. L. Neto[1], Michela Mulas[1] and Francesco Corona[1,2]

***Abstract*— In this work, the full-state controllability properties of a biological wastewater treatment plant are analysed. Specifically, the five biological reactors in the Benchmark Simulation Model no. 1 are studied. For the task, we represented the activated sludge plant as a dynamical system consisting of 65 states, 7 controls and 27 disturbances as a complex network and we studied its controllability properties from a classical and a structural point of view. By analysing the topology of the network, we show how the system is controllable in the structural sense and thus how it is controllable in the classical sense for almost all realisations of its parameters. Interestingly, we also found that the linearisation around a fixed point commonly used in the literature leads to a realisation of the system that is not full-state controllable in the classical sense. We show how this realisation is controllable if the state variables associated with inert matter are not considered. We quantify the compound and individual control efforts of this reduced-order system in terms of energy-based controllability metrics.**

***Index Terms*— Environmental systems, activated sludge process, controllability, structural control, complex networks.**

I. INTRODUCTION

Wastewater treatment systems are facing unprecedented challenges due to stricter effluent requirements, costs minimisation, sustainable reuse of water, nutrients and other resources, as well as the increasing expectation in the public to attain high service standards. Because of their wide diffusion, activated sludge processes play a key role in the biological treatment of wastewater and their efficient operation and control has a large technological and societal importance.

Many control strategies for activated sludge plants have been proposed in the industrial and academic literature, [1]. Large research efforts have been developed by using support tools that establish a simulation protocol for real activated sludge processes. Among them, the Benchmark Simulation Model no. 1 (BSM1, [2]), the simulation protocol and general platform for common activated sludge processes subjected to typical municipal wastewater influents, has permitted the design of a number of modelling and control solutions. However, little has been done to analyse the dynamics of this system, possibly because of the high-dimensionality of the state vector and the numerous steady-state conditions. To the best of our knowledge, only few studies (see [3], [4]) discuss state estimation and observability of BSMs analytically.

In this work, the controllability properties of a class of activated sludge plants represented by the BSM1 are investigated. For the task, we mapped the dynamical system consisting of 65 state variables, 7 controls and 27 disturbances onto a complex network and we studied its full-state controllability properties from a classical and a structural point of view. As we are primarily interested in determining whether the plant is controllable under all feasible linearisations, we studied the structural controllability ([5], [6]) of the model. According to our results, BSM1-plants are structurally controllable and thus they are controllable also for almost all linearisations. Structural controllability is also used to show that a linearisation commonly used in the literature is not full-state controllable, a result that is confirmed by the classical Popov-Belevitch-Hautus test [7]. We complete this work by reporting on compound energy-based controllability metrics and individual control efforts ([8], [9]). The last analysis refers to a reduced-order system of 55 state variables in which variables corresponding to inert matter are removed because decoupled or unreachable by the control variables.

The presentation is organised as follows: Section II describes a activated sludge plant and associated state-space model, Section III overviews the classical and structural notion of full-state controllability, Section IV reports and discusses our results about the full-state controllability and the control-energy metrics for this class of activated sludge plants.

II. THE ACTIVATED SLUDGE PLANT

We consider the activated sludge process in a conventional wastewater treatment plant. The process consists of five sequential biological reactors and a secondary settler (Fig. 1).

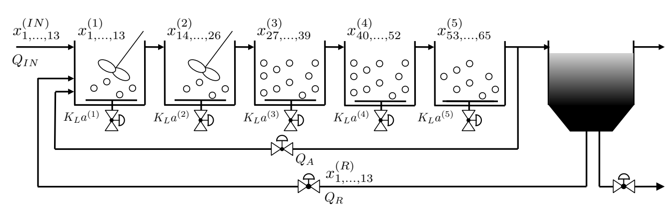

Fig. 1. The activated sludge plant.

The treatment is based on the denitrification-nitrification process in which bacteria reduce nitrogen present in form of nitrate and ammonia in the wastewater into nitrogen gas to be released into the atmosphere. No chemicals are added to the process and only oxygen is potentially added by insufflating air into each reactor. In the aerated reactors, the ammonium nitrogen (NH_4-N) contained in the wastewater is oxidised into nitrate nitrogen (NO_3-N), which is in turn

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reduced into nitrogen gas in the anoxic reactors. The process begins with a first reactor where wastewater from primary sedimentation, return sludge from secondary sedimentation and internal recycle sludge are fed. The outflow from the first reactor is then sequentially fed to the downstream reactors and, eventually, from the fifth reactor to the secondary settler. Mixed liquor from the fifth reactor is recirculated into the first reactor together with the recycle sludge from secondary sedimentation, as mentioned. Each reactor is described by the Activated Sludge Model no. 1 [10], while the settler is represented by the non-reactive model proposed in [11]. The bioprocess thus corresponds to the Benchmark Simulation Model no. 1 [2], here referred to as the activated sludge plant.

TABLE I

ACTIVATED SLUDGE PLANT: WASTEWATER CONCENTRATIONS, AS STATE VARIABLES ($k \in \{1, \dots, 5\}$) AND AS DISTURBANCES ($k = \{IN, R\}$).

State	Description	Units
$S_I^{(k)}$	Soluble inert organic matter	g COD m^{-3}
$S_S^{(k)}$	Readily biodegradable substrate	g COD m^{-3}
$X_I^{(k)}$	Particulate inert organic matter	g COD m^{-3}
$X_S^{(k)}$	Slowly biodegradable substrate	g COD m^{-3}
$X_{BH}^{(k)}$	Active heterotrophic biomass	g COD m^{-3}
$X_{BA}^{(k)}$	Active autotrophic biomass	g COD m^{-3}
$X_P^{(k)}$	Particulate products from biomass decay	g COD m^{-3}
$S_O^{(k)}$	Dissolved oxygen	g O_2 m^{-3}
$S_{NO}^{(k)}$	Nitrate and nitrite nitrogen	g N m^{-3}
$S_{NH}^{(k)}$	NH_4^+ + NH_3 nitrogen	g N m^{-3}
$S_{ND}^{(k)}$	Soluble biodegradable organic nitrogen	g N m^{-3}
$X_{ND}^{(k)}$	Particulate biodegradable organic nitrogen	g N m^{-3}
$S_{ALK}^{(k)}$	Alkalinity	mol HCO_3^- m^{-3}

From a system analytical perspective, the dynamics of each reactor in the activated sludge plant, if studied individually, are represented in terms of 13 state variables, the concentrations

$$x^{(k)} = \big[S_I^{(k)}\ S_S^{(k)}\ X_I^{(k)}\ X_S^{(k)}\ X_{BH}^{(k)}\ X_{BA}^{(k)}\ X_P^{(k)}\ S_O^{(k)}\ S_{NO}^{(k)}\ S_{NH}^{(k)}\ S_{ND}^{(k)}\ X_{ND}^{(k)}\ S_{ALK}^{(k)}\big],$$

and one controllable input, the oxygen transfer coefficient $u^{(k)} = K_La^{(k)}$. Moreover, the activated sludge plant is subjected to two additional controllable inputs, the internal and external sludge recycle flow-rates (Q_A and Q_R, respectively), and to 27 uncontrollable inputs or disturbances, the influent flow-rate Q_{IN} and concentrations $x^{(IN)}$, as well as the concentrations in the externally recirculated sludge, $x^{(R)}$, all directly implemented in the first reactor. We refer to Table I for a description of the concentration variables. The resulting state-space model of the activated sludge plant is

$$\dot{x}(t) = f(x(t), u(t), w(t)|\theta_x), \tag{1}$$

with state variables $x(t) \in \mathbb{R}_{\geq 0}^{N_x} = [x^{(1)} \ \cdots \ x^{(5)}]^T$, controllable inputs $u(t) \in \mathbb{R}_{\geq 0}^{N_u} = [Q_A \ Q_R \ K_La^{(1)} \ \cdots \ K_La^{(5)}]^T$, uncontrollable inputs $w(t) \in \mathbb{R}_{\geq 0}^{N_w} = [Q_{IN} \ x^{(IN)} \ x^{(R)}]^T$, and time-invariant dynamics $f(\cdot|\theta_x)$ depending on set of stoichiometric and kinetic parameters θ_x (see, [2]). For the state variables, we have $N_x = 13 \times 5 = 65$, $N_u = 2 + 5 = 7$ controllable inputs and $N_w = 1 + 13 + 13 = 27$ disturbances.

The default control strategy proposed for the BSM1 consists of two low-level controllers: $i)$ nitrate and nitrite nitrogen concentration in the second reactor, $S_{NO}^{(2)}$, by manipulation of the internal recycle Q_A; $ii)$ dissolved oxygen concentration in the fifth reactor, $S_O^{(5)}$, by manipulation of the oxygen mass transfer coefficient K_La. On a higher level, the performance of the plant is assessed in terms of flow-weighted time-averaged effluent concentrations of total suspended solids (TSS), biochemical oxygen demand (BOD_5), chemical oxygen demand (COD), total nitrogen (N_{TOT}) and ammonia (S_{NH}). Conventionally, control performance is assessed in terms of effluent quality by measuring and minimising the levels of these compound effluent concentrations. As these variables are defined as positive linear combinations of some state variables, their minimisation is achievable by minimising the component state variables. Our focus is on the full-state controllability of this class of activated sludge plants, we thus omit the measurement process $y(t) = g(x(t), u(t), w(t)|\theta_y)$.

III. PRELIMINARIES

For a dynamical system, the state-space representation

$$\dot{x}(t) = f_t(x(t), u(t), w(t)|\theta_x) \tag{2a}$$
$$y(t) = g_t(x(t), u(t), w(t)|\theta_y) \tag{2b}$$

describes how the state vector $x(t) \in \mathbb{R}^{N_x}$ evolves in time, given its current value and a set of controllable and uncontrollable but measurable input vectors $u(t) \in \mathbb{R}^{N_u}$ and $w(t) \in \mathbb{R}^{N_w}$ - the state equation Eq. (2a) - and how the state vector is emitted to form the measurement vector $y(t) \in \mathbb{R}^{N_y}$ - the measurement equation Eq. (2b). The nonlinear, time-varying and parametric vector functions $f_t(\cdot|\theta_x)$ and $g_t(\cdot|\theta_y)$ define the dynamics and the measurement process of the system, respectively. θ_x and θ_y are the model's parameters. In the following, we will limit ourselves to time-invariant systems $f(\cdot)$ and $g(\cdot)$, with a fully measurable state vector and without feedthrough of the inputs, $y(t) = Ix(t)$. With no loss of generality we will also assume an initial time $t_0 = 0$.

How state components interact with each other is captured by a $N_x \times N_x$ matrix A, whereas a $N_x \times N_u$ matrix B is used to identify which state components are affected by the controls and a $N_x \times N_w$ matrix G can be used to identify which state components are affected by the disturbances,

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t). \tag{3}$$

The structure of matrix A, B and G can be determined using inference diagrams in such a way that element $A_{i,j}$ (respectively, $B_{i,j}$ and $G_{i,j}$) is non-zero and potentially unknown whenever component x_i (u_i and w_i) appears in the vector field $f_j(\cdot)$; that is, whenever the (i,j)-th element $\partial f_j/\partial x_i$ ($\partial f_j/\partial u_i$ and $\partial f_j/\partial w_i$) in the Jacobian matrix(es) is not identically null. A quantification of the strength of the interactions can be obtained when the Jacobians are evaluated at some specific point (x', u', w'); that is, for an associated linearised systems in which A, B and G are assumed to be known. Typically, a fixed-point is chosen for the linearisation.

A dynamical system is said to be controllable if it is possible to steer its state vector from any initial value to any final value, in finite time. This notion of controllability is, in general, a prerequisite for control and, for known linear time-invariant systems, sufficient and necessary controllability conditions have been derived from the classical definition:

Definition 1 *(Controllability, [12]). The pair (A, B) is said to be controllable if, given any initial state $x(0)$ and any final state $x(t_f)$, it is possible to design an input $u(t)$ that transfers $x(0)$ to $x(t_f)$ in finite time, i.e., for $0 < t_f < \infty$.*

However, when matrix A and B are known only structurally, we have to resort to the alternative notion of structural controllability and associated sufficient and necessary conditions. In this section, we briefly overview the two approaches.

A. Classical controllability

Let $W_c(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} \mathrm{d}\tau$ be the $N_x \times N_x$ controllability Gramian of a system (A, B), a sufficient and necessary condition for controllability is that $\mathtt{det}(W_c(t_0, t)) \neq 0$ for any $t > t_0$. Though this criterion allows for a straightforward determination of the control from $x(0)$ to $x(t_f)$ of minimum quadratic effort or control-energy $E(t) \equiv \int_0^t \|u(t)\|^2 \mathrm{d}t$, its computation is unpractical. Equivalently, let $\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{N_x-1}B \end{bmatrix}$ be the $\mathbb{R}^{N_x \times N_x N_u}$ system's controllability matrix, a sufficient and necessary condition for controllability is that $\mathtt{rank}(\mathcal{C}) = N_x$, thus $\mathcal{C}$ must be full-rank [12]. This criterion is more direct and, for low-dimensional systems, its evaluation only requires a small number of matrix multiplications. However, when the dimensionality of the state vector is large, the computation of matrix $\mathcal{C}$ is troublesome because numerically ill-posed.

A scalable alternative is given by the Popov-Belevitch-Hautus (PBH) controllability test, based on the Hautus lemma:

Lemma 1 *(Hautus, [7]). Let $\sigma(A) = \{\lambda_i\}_{i=1}^{N_x}$ be the spectrum of A. The statement 'the pair (A, B) is controllable' is equivalent to the following statements:*

- $\mathtt{rank}\left(\begin{bmatrix} \lambda I - A & B \end{bmatrix}\right) = N_x$, *for all* $\lambda \in \mathbb{C}$;
- $\mathtt{rank}\left(\begin{bmatrix} \lambda_i I - A & B \end{bmatrix}\right) = N_x$, *for all* $\lambda_i \in \sigma(A) \subset \mathbb{C}$.

Thus, the pair (A, B) is controllable if and only if, for each eigenvalue λ_i of A (that is, when $\mathtt{rank}(\lambda_i I - A) < N_x$), the columns of B have at least one component in the direction $\nu_i \in \mathbb{R}^{N_x}$, ν_i being the eigenvector of A associated to λ_i. Moreover, eigenvectors ν_i for which $\mathtt{rank}([\lambda_i I - A\ B]) < N_x$ indicate directions in the state-space that are uncontrollable through the controls determined by matrix B.

Controllability tests are characterised by their binary nature. A quantification of controllability can be derived from $E(t) = \left(x(t_f) - e^{At_f}x(0)\right)^T W_c^{-1}(t_f) \left(x(t_f) - e^{At_f}x(0)\right)$, the control-energy associated with the control of minimum effort $u(t) = B^T e^{A^T(t_f - t)} W_c^{-1}(t_f) \left(x(t_f) - e^{At_f}x(0)\right)$. The eigenvectors $\{\nu_i(\lambda_i)\}$ associated with the eigenvalues $\lambda_i \in \sigma(W_c(t_f))$ correspond to directions in the state space that are ever harder to control the smaller the eigenvalue. For stable systems, the infinite-horizon controllability Gramian,

$$W_c(\infty) = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau, \tag{4}$$

always exists and it can be computed efficiently by solving the Lyapunov equation $AW_c(\infty) + W_c(\infty)A^T + BB^T = 0$.

Definition 2 *(Energy-related controllability metrics, [8], [9]). Let $W_c(\infty)$ be the solution of $AW_c(\infty) + W_c(\infty)A^T + BB^T = 0$. The control effort associated with the pair (A, B) can be quantified according to the following scalar metrics:*

I. $\mathtt{trace}(W_c(\infty))$: *It is inversely related to the control effort averaged over all directions in the state-space;*

II. $\mathtt{trace}\left(W_c^\dagger(\infty)\right)$: *It is related to the control effort averaged over all directions in the state-space.*

III. $\log\left(\mathtt{det}(W_c(\infty))\right)$: *It is related to the volume of a N_x-dimensional hyper-ellipsoid whose points are reacheable with one unit or less of control energy.*

IV. $\lambda_{min}(W_c(\infty))$: *It is inversely related to the control energy along the least controllable eigen-direction.*

The control effort associated to attempting to control the full-state by only controlling one individual state variable x_i at a time is quantified by the average controllability centrality,

$$C_{AC}(i) = \mathtt{trace}\left(W_{c,i}(\infty)\right). \tag{5}$$

This non-negative quantity is computed by assuming a single control that actuates only on the i-th state variable: That is, when $B = e_i$, the unit vector of the standard basis of $\mathbb{R}^{N_x}$. The infinite-horizon controllability Gramians $W_{c,i}(\infty) \in \mathbb{R}^{N_x \times N_x}$ are computed independently for all $i \in \{1, \ldots, N_x\}$ as solution to $AW_{c,i}(\infty) + W_{c,i}(\infty)A^T + e_i e_i^T = 0$ (see [9]).

B. Structural controllability

The dynamics of a linear time-invariant system (A, B) can be studied by mapping the state equation onto a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The vertex set $\mathcal{V} = \mathcal{V}_A \cup \mathcal{V}_B$ consists of the union of vertex set $\mathcal{V}_A = \{x_1, \ldots, x_{N_x}\}$ of state components and of vertex set $\mathcal{V}_B = \{u_1, \ldots, u_{N_u}\}$ of controls, while the edge set $\mathcal{E} = \mathcal{E}_A \cup \mathcal{E}_B$ is the union of set $\mathcal{E}_A = \{(x_j, x_i) \mid A_{i,j} \neq 0\}$ of directed edges between state component vertices and set $\mathcal{E}_B = \{(u_k, x_i) \mid B_{i,k} \neq 0\}$ of directed edges between control vertices and state components vertices. If the elements of A and B are either zeros or unknown, then the system is referred to as a structured dynamical system [13].

The pair (A, B) is said to be structurally controllable if the nonzero elements of A and B can be set in such a way that the system is controllable in the classical sense. Formally,

Definition 3 *(Structural Controllability, [5]). The pair (A, B) is said to be structurally controllable if and only if there exists a controllable pair $(\bar{A}, \bar{B})$ of the same dimension and structure of the pair (A, B) such that $\|\bar{A} - A\| < \varepsilon$ and $\|\bar{B} - B\| < \varepsilon$, for an arbitrary small $\varepsilon > 0$.*

Two pairs (A, B) and $(\bar{A}, \bar{B})$ have the same structure if they have the same dimensions and each element $A_{i,j} \neq 0$ (respectively, $B_{i,j} \neq 0$) whenever $\bar{A}_{i,j} \neq 0$ ($\bar{B}_{i,j} \neq 0$).

Lemma 2 *([5]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the directed network associated to the pair (A, B). (A, B) is said to be structurally controllable if and only if the following conditions hold:*

- *(Accessibility) For every $x_i \in \mathcal{V}_A$ there exists at least one directed path starting from any $u_k \in \mathcal{V}_B$ to x_i.*
- *(Dilation-free) For every $\mathcal{S} \subseteq \mathcal{V}_A$, $|T(\mathcal{S})| \geq |\mathcal{S}|$, where $T(\mathcal{S}) = \{x_j \in \mathcal{V} \mid x_i \in \mathcal{S} \wedge (x_j, x_i) \in \mathcal{E}\}$ denotes the neighborhood set of $\mathcal{S}$.*

The first condition can be verified by identifying all state component vertices that are accessible from each possible origin vertex (a control): Any graph search algorithm can be used for the task [14]. The second condition can be verified by computing a maximum matching $\mathcal{M} \subseteq \Gamma$ of an equivalent bipartite graph $\mathcal{K} = (\mathcal{V}_A^+ \cup \mathcal{V}_A^-, \Gamma)$ and then checking that all unmatched state vertices $x_j \in V_A^-$ are directly connected by distinct control vertices [6]. The maximum matching problem consists of identifying the (possibly not unique) subset of edges without common vertices that has maximum cardinality. The bipartite graph $\mathcal{K} = (\mathcal{V}_A^+ \cup \mathcal{V}_A^-, \Gamma)$ is defined by the disjoint and independent vertex sets $\mathcal{V}_A^+ = \{x_1^+, \cdots, x_{N_x}^+\}$ and $\mathcal{V}_A^- = \{x_1^-, \cdots, x_{N_x}^-\}$, and by the undirected edge set $\Gamma = \{(x_i^+, x_j^-) \mid (x_i, x_j) \in \mathcal{E}\}$. Distinct origins linked to the unmatched vertices form a $\mathcal{V}_A^-$−perfect matching. Guarantee of the dilation-free condition follows from Hall theorem [15].

IV. Results and discussion

In this section, we present the results of the analysis of the controllability of the class of activated sludge process plants represented by model (1) - $\dot{x}(t) = f(x(t), u(t), w(t)|\theta_x)$ with $N_x = 65$, $N_u = 7$ and $N_w = 27$ - defined in Section II. We present the results about the structural controllability of the associated structural system (A, B) and then we discuss the results obtained for some common linearisation (A^{SS}, B^{SS}).

A. Structural controllability analysis

For the activated sludge plant $\dot{x}(t) = f(x(t), u(t), w(t)|\theta_x)$ with $N_x = 65$, $N_u = 7$ and $N_w = 27$, the structural pair (A, B) is obtained from the Jacobian matrices, in such a way that $A \in \mathbb{R}^{65\times 65} = \partial f/\partial x$ and $B \in \mathbb{R}^{65\times 7} = \partial f/\partial u$. The digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for pair (A, B) is defined by vertex set $\mathcal{V} = \mathcal{V}_A \cup \mathcal{V}_B = \{x_1, \cdots, x_{65}\} \cup \{u_1, \cdots, u_7\}$, and directed edge set $\mathcal{E} = \mathcal{E}_A \cup \mathcal{E}_B = \{(x_j, x_i) \mid A_{i,j} \neq 0\} \cup \{(u_k, x_i) \mid B_{i,k} \neq 0\}$. The network is depicted in Fig. 2.

The topology of network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ indicates that the pair (A, B) is structurally controllable, Lemma 2. The accessibility condition is satisfied because all state vertices are reachable from any control vertices. Specifically, it is easy to see how they all are reachable through one-edge paths starting from control vertices Q_A or Q_R. The dilation-free condition is satisfied through a perfect matching $\mathcal{M}$ of size $|\mathcal{M}| = N_x$ formed by choosing every state vertex's self-loop, thus leaving no vertex unmatched. A perfect matching such as $\mathcal{M}$ ensures the dilation-free condition and suggests that controls are only needed to ensure accessibility. This implies that the full-state could be controlled by manipulating a single control (for example, Q_A or Q_R) and then relying on the individual self-dynamics to reach the desirable state. Such control strategy is formally correct but clearly unviable in a real-world situation.

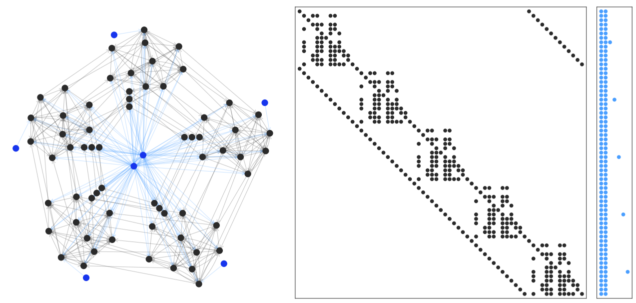

Fig. 2. Network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (left panel) associated to the structural pair (A, B) (middle and right panels). State vertices $x_i \in \mathcal{V}_A$ are in black, control vertices $u_k \in \mathcal{V}_B$ are in blue. State-state edges $(x_i, x_j) \in \mathcal{E}_A$ and control-state edges $(u_k, x_j) \in \mathcal{E}_B$ are coloured to match the corresponding entries in A and B. To avoid clutter, all state self-loops have been omitted.

Fig. 2 shows the existence of five subsets $\mathcal{S}_l \subseteq \mathcal{V}_A$ of state-vertices in which each pair of vertices $x_i, x_j \in \mathcal{S}_l$ is linked by a directed path along edges in $\mathcal{E}_A$. Subsets of vertices with such a property are known as strongly connected components (SCC). Importantly, the existence of a control vertex that actuates on any state vertex of each SCC is sufficient to ensure that all state component vertices are accessible. For the activated sludge plant, vertices associated with state components $S_I^{(1\rightsquigarrow 5)}$, $X_I^{(1\rightsquigarrow 5)}$, $X_P^{(1\rightsquigarrow 5)}$ and $S_{ALK}^{(1\rightsquigarrow 5)}$ in all reactors are associated to individual SCCs. The existence of such clusters of vertices reflects the fact that such variables are treated in model as non-reacting matter. Moreover, it is worth mentioning that the SCCs associated with $S_I^{(k)}$ and $X_I^{(k)}$ are disconnected from the other state components: These groups of states are characterised by decoupled dynamics.

Summarising, the plant $\dot{x}(t) = f(x(t), u(t), w(t)|\theta_x)$ is structurally controllable and thus it is also full-state controllable in a classical sense, for almost all possible realisations of A and B. Hence, it is possible to design a control $u(t)$ that is capable of transferring the plant to a desired state, in finite time, for almost any possible realisation (A, B). We conclude that the activated sludge plant is almost surely controllable for all possible linearisations of the model (that is, in the neighbourhood of all feasible operating points).

1) Linearisation around a commonly used operating-point: We consider the linearisation (A^{SS}, B^{SS}) around the fixed operating point $SS \equiv (x^{SS}, u^{SS}, w^{SS})$ considered in [2]. To verify whether the previous result holds for this realisation commonly used in the literature, the pair (A^{SS}, B^{SS}) is obtained from the Jacobian matrices instantiated at the equilibrium point (that is, $A^{SS} = \partial f/\partial x|_{SS}$ and $B^{SS} = \partial f/\partial u|_{SS}$). The, now weighted, digraph $\mathcal{G}_{SS} = (\mathcal{V}_{SS}, \mathcal{E}_{SS})$ is constructed accordingly with $\mathcal{V}_{SS} = \mathcal{V}_{A^{SS}} \cup \mathcal{V}_{B^{SS}} = \{x_1, \cdots, x_{65}\} \cup \{u_1, \cdots, u_7\}$, and directed edge set $\mathcal{E}_{SS} = \mathcal{E}_{A^{SS}} \cup \mathcal{E}_{B^{SS}} = \{(x_j, x_i) \mid A_{i,j}^{SS} \neq 0\} \cup \{(u_k, x_i) \mid B_{i,k}^{SS} \neq 0\}$.

Fig. 3 shows how the pair (A^{SS}, B^{SS}) is not full-state controllable in a structural sense. As the SCC comprising state components $S_I^{(1\rightsquigarrow 5)}$ cannot be reached from any of the control vertices, the accessibility condition is not satisfied.

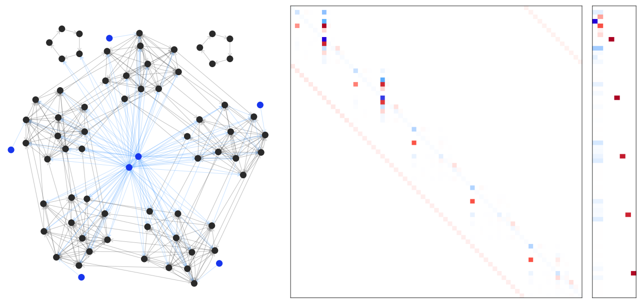

Fig. 3. Network $\mathcal{G}_{SS} = (\mathcal{V}_{SS}, \mathcal{E}_{SS})$ (left) associated to pair (A^{SS}, B^{SS}) (middle and right). State vertices $x_i \in \mathcal{V}_{A^{SS}}$ and state-state edges $(x_i, x_j) \in \mathcal{E}_{A^{SS}}$ are in black, control vertices $u_k \in \mathcal{V}_{B^{SS}}$ and control-state edges $(u_k, x_i) \in \mathcal{E}_{B^{SS}}$ are in blue. The groups of 5 vertices extruded from the network correspond to state components $X_I^{(1\rightsquigarrow 5)}$ and $S_I^{(1\rightsquigarrow 5)}$.

The emergence of inaccessible states is not specific to this linearisation. In particular, as $\{S_I^{(k)}\}_{k=1}^5$ represent soluble inert organic matter, their dynamics are modelled in (1) as $\{f_{13k+1} = \dot{S}_I^{(k)} = Q^{(k)}(S_I^{(k-1)} - S_I^{(k)})\}_{k=1}^5$, with $Q^{(k)}$ the flow-rate into the k-th reactor. Since we assume constant liquid volumes in the reactors, $\{Q^{(k)} = Q_A + Q_R + Q_{IN} = u_1 + u_2 + w_1\}_{k=1}^5$, steady-state $\{\dot{S}_I^{(k)} = 0\}_{k=1}^5$ can be achieved only when $S_I^{(k)}$ is the same in all reactors. Since

$$\frac{\partial f_{13k+1}}{\partial u_1}\Big|_{SS} = \frac{\partial f_{13k+1}}{\partial u_2}\Big|_{SS} = (S_I^{(k-1)} - S_I^{(k)}) = 0 \quad (\forall k),$$

for any linearisation around a steady-state, vertices $\{S_I^{(k)}\}_{k=1}^5$ will always be unaccessible from the control vertices.

A structurally controllable reduced-order system that does not include $\{S_I^{(k)}\}_{k=1}^5$ can be obtained by excluding the appropriate rows and columns in (A^{SS}, B^{SS}). In this case, while some vertices in the SCCs containing $X_I^{(k)}$ would disconnect from control origins, the SCC themselves would still be connected, thus guaranteeing accessibility to all state vertices. Moreover, given that the linearisation preserves every state vertex's self-loop, the dilation-free condition would still be satisfied by the perfect matching of choosing these edges.

B. Classical controllability analysis

As pair (A^{SS}, B^{SS}) corresponds to the linear time-invariant approximation of (1) in the neighbourhood of steady-state point SS, it is possible to analyse stability of A^{SS} and full-state controllability of such pair in the classical sense.

The spectrum $\sigma(A^{SS})$ consists of 34 distinct eigenvalues $\{\lambda_i(A^{SS})\}$, with $\{\lambda_1, \lambda_1^*, \cdots, \lambda_{16}, \lambda_{16}^*\} \subset \mathbb{C}$ and $\{\lambda_{17}, \cdots, \lambda_{34}\} \subset \mathbb{R}$. All eigenvalues are distinct, except for two complex conjugate pairs and one real value, each with algebraic multiplicity equal four. As $\texttt{Re}(\lambda_i) < 0$ for all $\lambda_i \in \sigma(A^{SS})$, then A^{SS} is a stable matrix and its spectrum thus characterises the plant as asymptotically stable.

We have shown that (A^{SS}, B^{SS}) is not a full-state controllable pair in a structural sense. This important result can be verified only using the PBH controllability test in Lemma (1), as an accurate computation of the controllability matrix is unfeasible. As expected, the PBH test confirms that the pair is not controllable, as the two complex conjugated pairs and one real eigenvalue, each with multiplicity equal four, lead to rank-deficient matrices $[\lambda_i I - A^{SS}\ B^{SS}]$. The twelve associated eigenvectors show the state-space directions that cannot be reached by using B^{SS}, Fig. 4. Specifically, three eigen-directions are defined only by inaccessible states $S_I^{(k)}$, and nine directions associate with states $S_I^{(k)}$, $X_I^{(k)}$, $X_P^{(k)}$ and $S_{ALK}^{(k)}$. Interestingly, these state variables associate with individual SCC in the networks previously discussed.

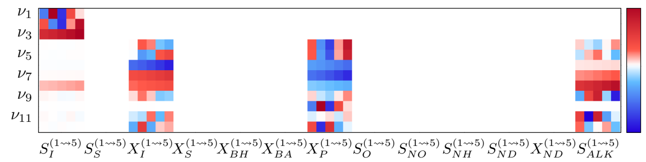

Fig. 4. Pair (A^{SS}, B^{SS}): Eigenvectors ν_i associated with eigenvalues $\lambda_i \in \sigma(A^{SS})$ that do not satisfy the PBH test ($\texttt{rank}([\lambda_i I - A^{SS}\ B^{SS}]) < 65$).

Conversely to what was obtained in structural terms, removing from (A^{SS}, B^{SS}) the entries associated with state components $\{S_I^{(k)}\}_{k=1}^5$ does not lead to a controllable reduced-order system. In fact, being $\{S_I^{(k)}\}$ characterised by decoupled dynamics, their exclusion only removes three uncontrollable eigen-directions. The apparent contradiction between structural and classical controllability can be explained from the dilation-free condition. The existence of a self-loop for each state vertex is sufficient to satisfy this condition, thus making that control vertices needed only to satisfy the accessibility condition. Whenever some of the self-loop weights are equal, the maximum matching will underestimate the number of controls needed for full-state controllability [16]. This is the case with (A^{SS}, B^{SS}), where $X_I^{(k)}$, $X_P^{(k)}$, and $S_{ALK}^{(k)}$ always have identical self-dynamics.

We further analysed the controllability of the reduced-order systems that do not include $X_I^{(k)}$ (respectively, $X_P^{(k)}$ or $S_{ALK}^{(k)}$) with $k \in \{1, \cdots 5\}$), together with the exclusion of $S_I^{(k)}$. Because representing non-reactive matter, the exclusion of these variables does not affect the other state variables. Each of the resulting systems has order 55 and is full-state controllable, according to the PBH controllability test.

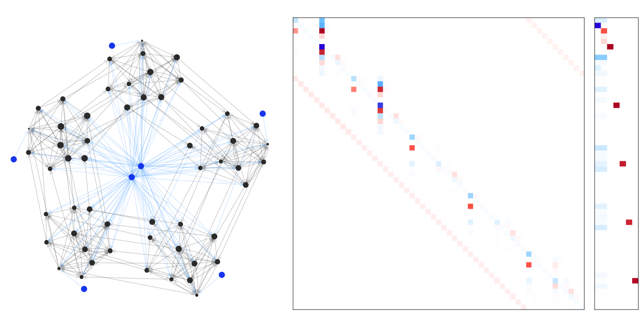

Fig. 5. Network $\widetilde{\mathcal{G}}_{SS} = (\widetilde{\mathcal{V}}_{SS}, \widetilde{\mathcal{E}}_{SS})$ (left) associated to pair $(\widetilde{A}^{SS}, \widetilde{B}^{SS})$ (middle and right). The size of state vertices is proportional to $C_{AC}(i)$.

One such reduced-order system $\widetilde{A}^{SS} \in \mathbb{R}^{55\times 55}$ and $\widetilde{B}^{SS} \in \mathbb{R}^{55\times 7}$ can be obtained by removing from (A^{SS}, B^{SS}) the entries relative to state components $\{S_I^{(k)}\}_{k=1}^5$ and

$\{X_I^{(k)}\}_{k=1}^5$. Removing these variables does not affect the dynamics of the remaining state components, since the SCCs are disconnected. The reduced-order system, Fig. 5, has a spectrum with 34 distinct eigenvalues $\lambda_i \in \sigma(\widetilde{A}^{SS})$ and two complex conjugate pairs and one real eigenvalue of multiplicity equal two. The system is stable (as we have that $\mathtt{Re}(\lambda_i(\widetilde{A}^{SS}))$ is always negative) and it is also full-state controllable in both the structural and classical sense.

We analysed the compound control effort in terms of the energy-related metrics, Definition 2, based on the infinite-horizon controllability Gramian $W_c(\infty)$ of $(\widetilde{A}^{SS}, \widetilde{B}^{SS})$, Eq. (4). The metrics, Table II, reveal how attempting to control even this reduced-order model is an energy demanding task. The fact that $\lambda_{min}(W_c(\infty))$ is virtually zero implies that there exists at least one state-space direction that is practically uncontrollable. Moreover, a very small eigenvalue causes $W_c(\infty)$ to have a large condition number and its determinant to be practically equal to zero ($\log(\mathtt{det}(W_c(\infty))) = -\infty$).

TABLE II

PAIR $(\widetilde{A}^{SS}, \widetilde{B}^{SS})$: ENERGY-RELATED CONTROLLABILITY METRICS.

$\mathtt{tr}(W_c(\infty))$	$\mathtt{tr}(W_c^\dagger(\infty))$	$\log(\mathtt{det}(W_c(\infty)))$	$\lambda_{\min}(W_c(\infty))$
0.8269	1.54E13	$-\infty$	−1.16E−17

We conclude that the pair $(\widetilde{A}^{SS}, \widetilde{B}^{SS})$, although controllable, it requires a very large control-effort to be able to access the full state-space ($W_c(\infty)$ is close to singular). The cumulative coverage of state-space is shown in Fig. 6 (left) in terms of normalised cumulative sum ($\Lambda(N) = \sum_{n=1}^{N} \lambda_n / \sum_{n_x=1}^{N_x} \lambda_{n_x}$) of the eigenvalues of $W_c(\infty)$, along with its eigenvectors.

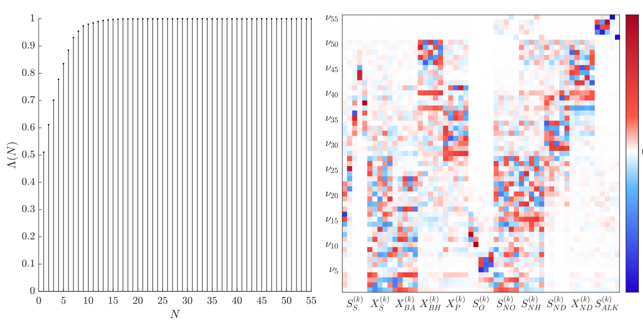

Fig. 6. Pair $(\widetilde{A}^{SS}, \widetilde{B}^{SS})$: Cumulative sum $\Lambda(N)$ of eigenvalues of the infinite-horizon controllability Gramian $W_c(\infty)$ and its eigenvectors (ν_n) .

The average energy that the system $(\widetilde{A}^{SS}, \widetilde{B}^{SS})$ would require if we were to control it by directly actuating only on one state variable at a time was quantified by the average controllability centrality $C_{AC}(i)$, Eq. (5), and the results depicted in Fig. 5. The analysis shows that the energy required to control the system is among the lowest if we were to actuate on some control that only affects biomass concentrations ($X_{BH}^{(k)}$ or $X_{BA}^{(k)}$). This reflects the fact that such variables are central to the process, but will evolve slowly if not controlled. Conversely, the energy required by the system would be the highest if we were to actuate on some control that only affects dissolved oxygen ($S_O^{(k)}$). However, it is worth mentioning that it is still possible to control $S_O^{(k)}$ (through $K_La^{(k)}$), while controlling $X_{BH}^{(k)}$ or $X_{BA}^{(k)}$ is practically unfeasible.

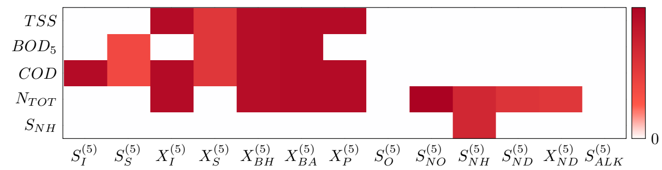

Fig. 7. Pair (A^{SS}, B^{SS}): Control effort $C_{AC}(i)$ associated to the state variables (x-axis) that contribute to define the controlled variables (y-axis).

We conclude with a discussion on how these control-efforts can be related to the measurement variables that are routinely controlled in the activated sludge plant. As the performance of the plant is quantified in terms of effluent concentrations, in Fig. 7 we show the control effort $C_{AC}(i)$ associated to the state variables that contribute to define TSS, BOD_5, COD, N_{TOT}, and S_{NH} in the effluent. To show the contribution from $S_I^{(5)}$ and $X_I^{(5)}$, the reported control centralities are based on the pair (A^{SS}, B^{SS}), being the results for the remaining state variables identical to what obtained for $(\widetilde{A}^{SS}, \widetilde{B}^{SS})$. As all these measurement and thus controllable variables depend on state variables that relate to biomass ($X_{BH}^{(5)}$, $X_{BA}^{(5)}$) and $X_P^{(5)}$, with the exception of S_{NH}, we conclude by pointing out that they also associate with high individual control efforts.

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